

Expected Supremum Representation of a Class of Single Boundary Stopping Problems

Luis H. R. Alvarez E.* Pekka Matomäki †
 Department of Accounting and Finance
 Turku School of Economics
 FIN-20014 University of Turku
 Finland

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Abstract

We consider the representation of the value of a class of optimal stopping problems of linear diffusions in a linearized form as an expected supremum of a known function. We establish an explicit integral representation of this representing function by utilizing the explicitly known marginals of the joint probability distribution of the extremal processes. We also delineate circumstances under which the value of a stopping problem induces directly this representation and show how it is connected with the monotonicity of the generator. We compare our findings with existing literature and show, for example, how our representation is linked to the smooth fit principle and how it coincides with the optimal stopping signal representation. The intricacies of the developed integral representation are explicitly illustrated in various examples arising in financial applications of optimal stopping.

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*luis.alvarez@tse.fi

†pjsila@utu.fi

1 Introduction

It is well-known from the literature on stochastic processes that the probability distributions of first hitting times are closely related to the probability distributions of the running supremum and running infimum of the underlying diffusion. Consequently, the question of whether a linear diffusion has exited from an open interval prior to a given date or not can be answered by studying the behavior of the extremal processes up to the date in question. If the extremal processes have remained in the open interval up to the particular date, then the process has not yet hit the boundaries and vice versa. In this study we utilize this connection and develop a linearized representation of the value function of an optimal stopping problem as the expected supremum of a representing function with known properties in the spirit of the pioneering work by [20, 21] and its subsequent extension to the treatment of optimal stopping problems by [11]. More formally, we plan to determine *explicitly* the nondecreasing, nonnegative, and upper semicontinuous representing function f for which

$$V(x) = \mathbb{E}_x [\sup\{f(X_t); t \leq T\}], \quad (1)$$

where $V(x)$ denotes the value of the considered class of optimal stopping problems and $T \sim \text{Exp}(r)$ is an exponentially distributed random time independent of the underlying process X .

The relatively recent literature on stochastic control theory indicates that the connection between, among others, the value functions and extremal processes in optimal stopping and singular stochastic control problems goes far beyond the standard connection between first hitting times and the running supremum and infimum of the underlying process (see, for example, [4, 5, 6, 7, 9, 14, 17, 15, 18, 19]). Essentially, in these studies the determination of the optimal policy and its value is shown to be equivalent with the existence of an appropriate optional projection involving the running supremum of a progressively measurable process (known as the *Bank - El Karoui representation*). The advantage of the representation utilized in these studies is that it is very general and applies also outside the standard Markovian and infinite horizon setting. Moreover, it can be utilized for studying and solving other stochastic control problems as well. For example, as was shown in [5, 6], the approach is applicable in the analysis of the Gittins-index familiar from the literature on multi-armed bandits (cf. [16, 22, 23, 24, 26]).

Instead of establishing directly how the value of the considered class of optimal stopping problems can be expressed as an expected supremum, we take an alternative route and compute first explicitly the expected value of the supremum of an unknown function satisfying a set of monotonicity and regularity conditions by utilizing the

known probability distribution of the running supremum of the underlying. Setting this expected value equal with the value of the optimal stopping problem then results into a functional identity from which the unknown function can be explicitly determined. In the considered single boundary setting the function admits a relatively simple characterization in terms of the increasing minimal excessive mapping for the underlying diffusion (cf. [4]). We find that the required monotonicity of the function needed for the representation is closely related with the monotonicity of the generator on the state space of the underlying process. However, since only the sign of the generator typically affects the determination of the optimal strategy and its value, our results demonstrate that not all single boundary problems can be represented as the expected supremum of a monotonic function. We also investigate the regularity properties of the function needed for the representation and show that it needs not be continuous at the optimal stopping boundary. More precisely, we find that if the optimal boundary is attained at a point where the exercise payoff is not differentiable and, hence, the standard smooth fit condition is not satisfied, then the representing function is only upper semicontinuous at the optimal boundary. This is a result which is in line with the findings by [11].

The contents of this study is as follows. In section two we formulate the considered problem, characterize the underlying stochastic dynamics, and state a set of auxiliary results needed in the subsequent analysis of the problem. Section three focuses on a single boundary setting where the optimal rule is to exercise as soon as a given exercise threshold is exceeded. Our general findings on the representing function are explicitly illustrated in section four in various settings including incentive compatible stopping rules, Gittins indices, optimal entry, and stopping of spectrally negative jump diffusions. Finally, section five concludes our study.

2 Problem Formulation

2.1 Underlying stochastic dynamics

We consider a linear, time homogeneous and regular diffusion process $X = \{X(t); t \in [0, \xi)\}$, where ξ denotes the possible infinite life time of the diffusion. We assume that the diffusion is defined on a complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F})$, and that the state space of the diffusion is $\mathcal{I} = (a, b) \subset \mathbb{R}$. Moreover, we assume that the diffusion does not die inside \mathcal{I} , implying that the boundaries a and b are either natural, entrance, exit or regular (see Section II. 1 in [8] for a characterization of the boundary behaviour of diffusions). If a boundary is regular, we assume that it is killing and that the process

X is immediately sent to a cemetery state $\partial \notin \mathcal{I}$ as soon as it hits that boundary. Furthermore we will denote by $M_t = \sup\{X_s; s \in [0, t]\}$ the running supremum process of the considered diffusion X_t .

As usually, we denote by \mathcal{A} the differential operator representing the infinitesimal generator of X . For a given smooth mapping $f : \mathcal{I} \mapsto \mathbb{R}$ this operator is given by

$$(\mathcal{A}f)(x) = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}f(x) + \mu(x)\frac{d}{dx}f(x),$$

where the drift coefficient $\mu : \mathcal{I} \mapsto \mathbb{R}$ and the volatility coefficient $\sigma : \mathcal{I} \mapsto \mathbb{R}_+$ are given continuous mappings. In order to avoid interior singularities, we assume throughout this study that $\sigma(x) > 0$ for all $x \in \mathcal{I}$. As is known from the classical theory on linear diffusions, there are two linearly independent *fundamental solutions* $\psi(x)$ and $\varphi(x)$ satisfying a set of appropriate boundary conditions based on the boundary behavior of the process X and spanning the set of solutions of the ordinary differential equation $(\mathcal{G}_r u)(x) = 0$, where $\mathcal{G}_r = \mathcal{A} - r$ denotes the differential operator associated with the diffusion X killed at the constant rate r . Moreover, $\psi'(x)\varphi(x) - \varphi'(x)\psi(x) = BS'(x)$, where $B > 0$ denotes the constant Wronskian of the fundamental solutions and

$$S'(x) = \exp\left(-\int^x \frac{2\mu(t)}{\sigma^2(t)} dt\right)$$

denotes the density of the scale function of X (for a comprehensive characterization of the fundamental solutions, see [8], pp. 18–19). The functions ψ and φ are minimal in the sense that any non-trivial r -excessive mapping for X can be expressed as a combination of these two (cf. [8], pp. 32–35). Given the fundamental solutions, let $u(x) = c_1\psi(x) + c_2\varphi(x)$, $c_1, c_2 \in \mathbb{R}$ be an arbitrary twice continuously differentiable r -harmonic function and define for sufficiently smooth mappings $g : \mathcal{I} \mapsto \mathbb{R}$ the functional

$$(L_u g)(x) = g(x)\frac{u'(x)}{S'(x)} - \frac{g'(x)}{S'(x)}u(x) = c_1(L_\psi g)(x) + c_2(L_\varphi g)(x)$$

associated with the representing measure for r -excessive functions (cf. [33]). Noticing that if g is twice continuously differentiable, then

$$(L_u g)'(x) = -(\mathcal{G}_r g)(x)u(x)m'(x) \quad (2)$$

where $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the density of the speed measure m of X . Hence, we find that

$$(L_u g)(y) - (L_u g)(z) = \int_y^z (\mathcal{G}_r g)(v)u(v)m'(v)dv \quad (3)$$

for any $a < y < z < b$.

Finally, we denote by $\mathcal{L}_r^1(\mathcal{I})$ the class of measurable functions $f : \mathcal{I} \mapsto \mathbb{R}_+$ satisfying the integrability condition

$$\mathbb{E}_x \int_0^\infty e^{-rs} |f(X_s)| ds < \infty$$

for all $x \in \mathcal{I}$. As is known from the literature on linear diffusions, if $f \in \mathcal{L}_r^1(\mathcal{I})$ then its expected cumulative present value

$$(R_r f)(x) = \mathbb{E}_x \int_0^\infty e^{-rs} f(X_s) ds$$

can be expressed as (cf. [8], p. 29)

$$(R_r f)(x) = \int_a^b G_r(x, v) f(v) m'(v) dv, \quad (4)$$

where

$$G_r(x, v) = \begin{cases} B^{-1} \varphi(v) \psi(x), & x \leq v, \\ B^{-1} \varphi(x) \psi(v), & x \geq v. \end{cases} \quad (5)$$

2.2 The Optimal Stopping Problem and Auxiliary Results

In this paper our objective is to examine the optimal stopping problem

$$V(x) = \sup_{\tau} \mathbb{E}_x [e^{-r\tau} g(X_\tau)] \quad (6)$$

for exercise payoff functions g satisfying a set of sufficient regularity conditions and establish a representation of the value V as the expected supremum of an appropriately chosen representing function along the lines of the pioneering studies [5], [6], [11], [15], [17], [20], [21]. Our main result is based on the following representation theorem originally established in [11].

Theorem 2.1. ([11], Theorem 2.5) *Let X_t be a Hunt process on \mathcal{I} and $T \sim \text{Exp}(r) \perp X_t$. Assume that the exercise payoff g is non-negative, lower semicontinuous, and satisfies the condition $\mathbb{E}_x [\sup_{t \geq 0} e^{-rt} g(X_t)] < \infty$ for all $x \in \mathcal{I}$. Assume also that there exists an upper semicontinuous \hat{f} and a point $y^* \in \mathcal{I}$ such that*

- (a) $\hat{f}(x) \leq 0$ for $x < y^*$, $\hat{f}(x)$ is non-decreasing and positive for $x \geq y^*$,
- (b) $\mathbb{E}_x [\sup_{0 \leq t \leq T} \hat{f}(X_t)] = g(x)$ for $x \geq y^*$, and

$$(c) \mathbb{E}_x \left[\sup_{0 \leq t \leq T} \hat{f}(X_t) \right] \geq g(x) \text{ for } x \leq y^*.$$

Then

$$V(x) = \mathbb{E}_x \left[\sup_{0 \leq t \leq T} \hat{f}(X_t) \mathbb{1}_{[y^*, b)}(X_t) \right] = \mathbb{E}_x \left[\hat{f}(M_T) \mathbb{1}_{[y^*, b)}(M_T) \right] \quad (7)$$

and $\tau^* = \inf\{t \geq 0 : X_t > y^*\}$ is an optimal stopping time.

This theorem essentially states that if we can find a representing function \hat{f} satisfying the required conditions (a)-(c), then the optimal stopping policy for (6) constitutes an one-sided threshold rule and its value can be expressed in a linearized form as an expected supremum attained at an independent exponential random time. As we will prove later in this paper, the reverse argument is also sometimes true: under certain circumstances the value of the optimal policy generates a continuous and monotone function \hat{f} for which the representation (7) is valid. However, as we will point out later in the case where the exercise reward can be expressed as an expected cumulative present value of a continuous flow, all single boundary stopping problems cannot be represented as proposed in Theorem 2.1.

Before proceeding in our analysis and the explicit identification of the representing function, we first establish two auxiliary lemmata needed in the analysis of the problem. Our first findings based on the known joint probability distribution of the underlying and its running supremum are summarized in the following.

Lemma 2.2. (A) If $h : \mathcal{I} \mapsto \mathbb{R}$ satisfies $h \in \mathcal{L}_r^1(\mathcal{I})$, then

$$\frac{1}{r} \mathbb{E}_x[h(X_T) | M_T \leq y] = \frac{(R_r h)(x) - (R_r h)(y) \frac{\psi(x)}{\psi(y)}}{1 - \frac{\psi(x)}{\psi(y)}} \quad (8)$$

and

$$\frac{1}{r} \mathbb{E}_x[h(X_T) | M_T = y] = \frac{S'(y)}{\psi'(y)} \int_a^y h(v) \psi(v) m'(v) dv \quad (9)$$

for all $x \in (a, y]$. Especially, if $h \in C(\mathcal{I}) \cap \mathcal{L}_r^1(\mathcal{I})$, then

$$\begin{aligned} \lim_{x \uparrow y} \frac{1}{r} \mathbb{E}_x[h(X_T) | M_T \leq y] &= (R_r h)(y) - (R_r h)'(y) \frac{\psi(y)}{\psi'(y)} \\ &= \frac{S'(y)}{\psi'(y)} \int_a^y h(v) \psi(v) m'(v) dv \end{aligned} \quad (10)$$

for all $y \in \mathcal{I}$.

(B) If $\mathcal{G}_r h \in C(\mathcal{I} \setminus \mathcal{P}) \cap \mathcal{L}_r^1(\mathcal{I})$, where $\mathcal{P} \in \mathcal{I}$ is a finite set of points, and $\lim_{x \downarrow a} (L_\psi h)(x) = 0$, then for all $x \in (a, y]$

$$\frac{1}{r} \mathbb{E}_x[(\mathcal{G}_r h)(X_T) | M_T = y] = - \frac{(L_\psi h)(y)}{(L_\psi \mathbb{1})(y)}, \quad (11)$$

where $\mathbb{1} = \mathbb{1}_{\mathcal{I}}(x)$.

Proof. See Appendix A. \square

Second, in order to characterize how increased volatility affects the representing function, we need to state conditions under which the sign of the impact of increased volatility on the Laplace transform of the first hitting time to a constant boundary can be unambiguously described. A set of sufficient conditions under which this sign is positive are now stated in the following.

Lemma 2.3. *If $\psi(x)$ is convex, then increased volatility increases or leaves unchanged the ratio $\psi(x)/\psi(z)$ for all $a < x \leq z < b$ and decreases or leaves unchanged the ratio $\psi'(x)/\psi(x)$ for all $x \in \mathcal{I}$.*

Proof. See Appendix B. \square

3 Representation as Expected Supremum

3.1 Problem Setting

Our main objective is now to delineate general circumstances under which the value of a one-sided threshold policy can be expressed as the expected supremum of a monotonic representing function and to identify that function explicitly. In what follows, we will focus on the case where the considered stopping policy can be characterized as a rule where the underlying process is stopped as soon as it exceeds a given constant threshold. The case where the single boundary stopping rule is to exercise as soon as the underlying falls below a given constant threshold is completely analogous and, therefore, left untreated.

Let $g : \mathcal{I} \mapsto \mathbb{R}$ be a continuous payoff function satisfying the condition $g^{-1}(\mathbb{R}_+) = (x_g, b) \neq \emptyset$ for some $x_g \in \mathcal{I}$ and

$$\mathbb{E}_x \left[\sup_{t \geq 0} e^{-rt} g(X_t) \right] < \infty \quad (12)$$

for all $x \in \mathcal{I}$. Assume also that $g \in C^1(\mathcal{I} \setminus \mathcal{P}) \cap C^2(\mathcal{I} \setminus \mathcal{P})$, where $\mathcal{P} \in \mathcal{I}$ is a finite set of points in \mathcal{I} and that $|g'(x_{\pm})| < \infty$ and $|g''(x_{\pm})| < \infty$ for all $x \in \mathcal{P}$.

Given the assumed regularity conditions, let $\tau_y = \inf\{t \geq 0 : X_t \geq y\}$ denote the first exit time of the underlying diffusion from the set (a, y) , where $y \in g^{-1}(\mathbb{R}_+)$. Define now the parameterized family of nonnegative and continuous functions $V_y : \mathcal{I} \mapsto \mathbb{R}_+$ by

$$V_y(x) = \mathbb{E}_x \left[e^{-r\tau_y} g(X_{\tau_y}); \tau_y < \infty \right] = \begin{cases} g(x) & x \geq y \\ \psi(x) \frac{g(y)}{\psi(y)} & x < y. \end{cases} \quad (13)$$

Given representation (13), we can now state our identification problem as follows.

Problem 3.1. (A) For a given $y \in g^{-1}(\mathbb{R}_+)$, does there exist a non-negative function $\hat{f} : \mathcal{I} \mapsto \mathbb{R}_+$ such that for all $x \in \mathcal{I}$ we would have

$$J_y(x) := \mathbb{E}_x \left[\hat{f}(M_T) \mathbb{1}_{[y,b)}(M_T) \right] = V_y(x). \quad (14)$$

(B) Under which conditions on the function \hat{f} and the threshold y we have

$$\hat{f}(M_T) \mathbb{1}_{[y,b)}(M_T) = \sup_{t \in [0, T]} \{ \hat{f}(X_t) \mathbb{1}_{[y,b)}(X_t) \}$$

and, consequently,

$$V(x) = V_y(x) = \mathbb{E}_x \left[\sup_{t \in [0, T]} \{ \hat{f}(X_t) \mathbb{1}_{[y,b)}(X_t) \} \right]. \quad (15)$$

It's worth emphasizing that Problem 3.1 is twofold. The first representation problem essentially asks if the expected value of the exercise payoff accrued at the first hitting time to a constant boundary can be expressed as the expected value of an yet unknown representing function \hat{f} at the running maximum of the underlying diffusion at an independent exponentially distributed date. The second question essentially asks when the function \hat{f} is such that the representation agrees with the general functional form utilized in Theorem 2.1 and in that way results into the value of the considered stopping problem. As we will later establish in this paper, the class of functions satisfying the first representation is strictly larger than the latter.

3.2 Standard Sufficiency Conditions

Before proceeding in the derivation of the representation as an expected supremum, we first need to characterize sufficient conditions under which the value V_y coincides with the value of the optimal stopping problem (6). To accomplish this, we follow the Martin boundary representation approach introduced in the pioneering study [33] (for associated results focusing precisely on single-boundary problems, see [13]) and establish the following result characterizing the optimal policy. We apply this result later for the identification of circumstances under which the value of the considered one-sided problem can be expressed as the expected supremum of a monotonic function.

Lemma 3.2. Assume that the following conditions are satisfied:

- (i) there exists a $y^* = \operatorname{argmax}\{g(x)/\psi(x)\} \in \mathcal{I}$,
- (ii) $(\mathcal{G}_r g)(x) \leq 0$ for all $x \in [y^*, b) \setminus \mathcal{P}$

(iii) $g'(x+) \leq g'(x-)$ for all $x \in [y^*, b) \cap \mathcal{P}$

Then $V(x) = V_{y^*}(x)$ and $\tau_{y^*} = \inf\{t \geq 0 : X_t \geq y^*\}$ is an optimal stopping time.

Proof. See Appendix C. \square

Remark 3.3. *It is at this point worth emphasizing that under the following slightly stricter assumptions there always exists a unique maximizing threshold $y^* = \operatorname{argmax}\{g(x)/\psi(x)\}$ and the conditions of Lemma 3.2 are satisfied (cf. Lemma 3.4 in [2]):*

- (A) $g^{-1}(\mathbb{R}_-) = (a, y_0)$, where $a < y_0 < b$, and b is unattainable for X ,
- (B) there exists a $\tilde{x} \in \mathcal{I}$ so that $(\mathcal{G}_r g)(x) \geq 0$ for all $x \in (a, \tilde{x}) \setminus \mathcal{P}$ and $(\mathcal{G}_r g)(x) < 0$ for all $x \in (\tilde{x}, b) \setminus \mathcal{P}$,
- (C) $g'(x+) \geq g'(x-)$ for all $x \in (a, \tilde{x}) \cap \mathcal{P}$ and $g'(x+) \leq g'(x-)$ for all $x \in [\tilde{x}, b) \cap \mathcal{P}$

These assumptions of Remark 3.3 are typically met in financial applications of optimal stopping. Note that these conditions do not impose monotonicity requirements on the behavior of the generator $\mathcal{G}_r g$ on $\mathcal{I} \setminus \mathcal{P}$ and only the sign of $\mathcal{G}_r g$ essentially counts.

3.3 Characterization of the Representing Function \hat{f}

Let $y \in g^{-1}(\mathbb{R}_+)$ be given. Utilizing the known distribution function of M yields (cf. [8], p. 26)

$$J_y(x) = \mathbb{E}_x \left[\hat{f}(M_T) \mathbb{1}_{[y, b)}(M_T) \right] = \psi(x) \int_{x \vee y}^b \hat{f}(z) \frac{\psi'(z)}{\psi^2(z)} dz.$$

Given this expression, it is now sufficient to find a function \hat{f} for which the identity $V_y(x) = J_y(x)$ holds for all $x \in \mathcal{I}$. This identity holds for $x \geq y$ provided that the *Volterra integral equation of the first kind*

$$\frac{g(x)}{\psi(x)} = \int_x^b \hat{f}(z) \frac{\psi'(z)}{\psi^2(z)} dz \quad (16)$$

is satisfied. Standard differentiation of identity (16) now shows that for all $x \in [y, b) \setminus \mathcal{P}$ we have

$$\hat{f}(x) = g(x) - \psi(x) \frac{g'(x)}{\psi'(x)}, \quad (17)$$

coinciding with the function ρ derived in [4] by relying on functional concavity arguments. Utilizing (3) demonstrates that this representing function can be alternatively be expressed as

$$\hat{f}(x) = \frac{(L_\psi g)(x)}{(L_\psi \mathbb{1})(x)}. \quad (18)$$

Consequently, if $\mathcal{G}_r g \in C(\mathcal{I} \setminus \mathcal{P}) \cap \mathcal{L}_r^1(\mathcal{I})$, and $\lim_{x \downarrow a} (L_\psi g)(x) = 0$, then according to Lemma 2.2 we have

$$\hat{f}(z) = -\frac{1}{r} \mathbb{E}_x[(\mathcal{G}_r g)(X_T) | M_T = z]$$

for all $z \in \mathcal{I}$. Our first representation result is now stated in the following theorem.

Theorem 3.4. *Fix $y \in g^{-1}(\mathbb{R}_+)$ and let \hat{f} be as in (17). Then, if $\lim_{x \rightarrow b-} g(x)/\psi(x) = 0$, we have $J_y(x) = V_y(x)$. Moreover, if $\hat{f}(x)$ is also nonnegative and nondecreasing and $g'(x)$ is lower semicontinuous for all $x \in [y, b)$, then $V_y(x)$ is r -excessive for X .*

Proof. The first claim follows directly from identity (16) after noticing that the representing function can be re-expressed for all $x \in \mathcal{I} \setminus \mathcal{P}$ as

$$\hat{f}(x) = -\frac{\psi^2(x)}{\psi'(x)} \frac{d}{dx} \left(\frac{g(x)}{\psi(x)} \right)$$

and invoking the condition $\lim_{x \rightarrow b-} g(x)/\psi(x) = 0$. Noticing that since g, ψ , and ψ' are continuous the lower semicontinuity of g' on $[y, b)$ guarantees that \hat{f} is upper semicontinuous on $[y, b)$ as well. If \hat{f} is also nonnegative and nondecreasing, then $\hat{f}(x) \mathbb{1}_{[y, b)}(x)$ is nondecreasing, nonnegative, and upper semicontinuous on \mathcal{I} . In that case $\hat{f}(M_T) \mathbb{1}_{[y, b)}(M_T) = \sup_{t \in [0, T]} \{\hat{f}(X_t) \mathbb{1}_{[y, b)}(X_t)\}$ and Proposition 2.1 in [20] then guarantees that $J_y(x)$ is r -excessive for X . Since $J_y(x) = V_y(x)$ the alleged result follows. \square

Theorem 3.4 shows that when \hat{f} is chosen according to the rule (17) representation $J_y = V_y$ is valid provided that the limiting condition $\lim_{x \rightarrow b-} g(x)/\psi(x) = 0$ is met. Moreover, Theorem 3.4 also shows that if $\hat{f}(x) \mathbb{1}_{[y, b)}(x)$ is also nonnegative, nondecreasing, and upper semicontinuous, then the representation is r -excessive for the underlying diffusion X . This observation is of interest since it demonstrates that the needed monotonicity of the representing function does not, in principle, require twice differentiability of the exercise payoff g . This is especially beneficial in the verification of the r -excessivity of a value since it essentially reduces the analysis into the analysis of the sign, monotonicity and semicontinuity of \hat{f} . Note, however, that the

representation needs not to majorize the exercise payoff and, therefore, it does not necessarily coincide with the value of the considered stopping problem. Moreover, the required conditions for $\hat{f}(x)\mathbb{1}_{[y,b)}(x)$ are sufficient but *not necessary* for the r -excessivity of J_y . As we will later see, there are circumstances where J_y is r -excessive even when $\hat{f}(x)\mathbb{1}_{[y,b)}(x)$ is not monotonic.

An interesting comparative static result characterizing the sign of the relationship between increased volatility and the representing function \hat{f} defined by (17) is now summarized in the next theorem.

Theorem 3.5. *Assume that the exercise reward g is nondecreasing and that ψ is convex. Then, increased volatility decreases or leaves unchanged the value of the representing function \hat{f} defined by (17). Moreover, increased volatility increases the expected value $\mathbb{E}_x[f(M_T)]$ for nondecreasing functions $f : \mathcal{I} \mapsto \mathbb{R}_+$ satisfying $f \in \mathcal{L}_r^1(\mathcal{I})$.*

Proof. As shown in Lemma 2.3, the assumed convexity of ψ guarantees that increased volatility decreases the logarithmic growth rate ψ'/ψ . Consequently, if the reward g is nondecreasing on \mathcal{I} , then increased volatility increases the product $g'\psi/\psi'$ for all $x \in \mathcal{I} \setminus \mathcal{P}$. However, the assumed monotonicity of g and the existence of the left- and right-hand limits at all $x \in \mathcal{P}$ demonstrates that increased volatility increases the product $g'(x\pm)\psi(x)/\psi'(x)$ for all $x \in \mathcal{P}$ as well. Applying this finding to the definition (17) of the representing function \hat{f} proves the first claim. On the other hand, utilizing the assumed monotonicity of the function f in connection with Fubini's theorem yields

$$\mathbb{E}_x[f(M_T)] = f(x) + \int_x^b \mathbb{P}_x[M_T \geq v] df(v) = f(x) + \int_x^b \frac{\psi(x)}{\psi(v)} df(v)$$

from which the alleged comparative static result follows. \square

Theorem 3.5 characterizes circumstances under which increased volatility unambiguously decreases or leaves unchanged the representing function \hat{f} and increases or leaves unchanged the expected value of nondecreasing functions depending on the running supremum M . As we will later observe, both of these results have economically interesting consequences.

Having characterized the basic properties of the representing function \hat{f} , we are now in position to establish the following theorem connecting the representing function approach to standard sufficiency conditions.

Theorem 3.6. *Assume that the conditions of Lemma 3.2 are satisfied, that $\lim_{x \rightarrow b} g(x)/\psi(x) = 0$, and that $g'(x)$ is lower semicontinuous on $[y^*, b)$. Then,*

$$V(x) = V_{y^*}(x) = J_{y^*}(x) = \mathbb{E}_x \left[\hat{f}(M_T) \mathbb{1}_{[y^*, b)}(M_T) \right].$$

Proof. It is clear that the conditions of the first claim of Theorem 3.4 are satisfied. Consequently, $J_{y^*}(x) = V_{y^*}(x)$. The alleged result now follows from Lemma 3.2. \square

Theorem 3.6 states a set of conditions under which the value of the optimal stopping strategy can be expressed as the expected value of the mapping \hat{f} at the running maximum of the underlying diffusion. However, this does not yet guarantee that the value of the stopping problem could be expressed as an expected supremum since that requires in addition to the conditions of Theorem 3.6 the monotonicity of \hat{f} . Moreover, Theorem 3.6 relies on a set of sufficiency conditions based on the sign of $\mathcal{G}_r g$ and as such utilizes second order properties of the exercise payoff. Hence, it is of interest to investigate if at least part of the assumptions could be relaxed in the verification of optimality and the validity of the representation as an expected supremum. A set of sufficient conditions resulting into the desired outcome are summarized in our next theorem.

Theorem 3.7. *Assume that the exercise payoff g is nondecreasing, that there is a unique interior threshold $y^* \in \mathcal{I}$ so that $\hat{f}(x) \leq 0$ for $x \in (a, y^*)$ and $\hat{f}(x) > 0$ for $x \in (y^*, b)$, that g' is lower semicontinuous on $[y^*, b)$, that \hat{f} is nondecreasing on $[y^*, b)$, and that $g(x)/\psi(x) \downarrow 0$ as $x \uparrow b$. Then,*

$$V(x) = V_{y^*}(x) = J_{y^*}(x) = \mathbb{E}_x \left[\sup_{t \in [0, T]} \{ \hat{f}(X_t) \mathbb{1}_{[y^*, b)}(X_t) \} \right] \quad (19)$$

for all $x \in \mathcal{I}$.

Proof. Since

$$\frac{d}{dx} \left(\frac{g(x)}{\psi(x)} \right) = - \frac{\psi'(x)}{\psi^2(x)} \hat{f}(x)$$

for all $x \in \mathcal{I} \setminus \mathcal{P}$, we notice that our assumptions on the sign of \hat{f} guarantee that $g(x)/\psi(x)$ is increasing on (a, y^*) and decreasing on (y^*, b) . Consequently, $y^* = \operatorname{argmax}\{g(x)/\psi(x)\}$ and $y^* \in \{x \in \mathcal{I} : V(x) = g(x)\}$ by Theorem 2.1 in [12]. The monotonicity of g and positivity of \hat{f} on (y^*, b) then imply that $y^* \in g^{-1}(\mathbb{R}_+)$. Since $g(x)/\psi(x) \downarrow 0$ as $x \uparrow b$, we find by utilizing Theorem 3.4 that $V_{y^*}(x) = J_{y^*}(x)$ for all $x \in \mathcal{I}$. Moreover, the lower semicontinuity of g' and monotonicity and positivity of \hat{f} on (y^*, b) guarantee that the conditions of the second claim of Theorem 3.4 are satisfied and, therefore, that $V_{y^*} = J_{y^*}$ is r -excessive for X . Since V_{y^*} majorizes the payoff g for all $x \in \mathcal{I}$ and V_{y^*} can be attained by utilizing the stopping strategy τ_{y^*} we notice that $V = V_{y^*} = J_{y^*}$. Finally the monotonicity of \hat{f} implies that $\hat{f}(M_T) \mathbb{1}_{[y^*, b)}(M_T) = \sup_{t \in [0, T]} \{ \hat{f}(X_t) \mathbb{1}_{[y^*, b)}(X_t) \}$ from which the alleged identity follows. \square

Theorem 3.7 states a set of conditions under which the value of the considered stopping problem admits a representation as an expected supremum. Instead of having to rely on the behavior of $\mathcal{G}_r g$, Theorem 3.7 demonstrates that the verification of the optimality of a single boundary stopping strategy can be reduced to the study of the sing, monotonicity and sufficient regularity of the representing function \hat{f} . This observation is very useful especially in situations where the fundamental solution ψ has a simple functional form since under such circumstances the verification of the validity of the conditions of Theorem 3.7 is straightforward. However, as soon as ψ takes more complicated forms, establishing the monotonicity of \hat{f} becomes significantly more challenging and requires further analysis. In order to characterize relatively general circumstances under which the function \hat{f} is indeed monotonic, we first state the following auxiliary lemma.

Lemma 3.8. *Let $y \in g^{-1}(\mathbb{R}_+)$ be given. Assume that either*

- (A) *$g(x)$ is concave and $\psi(x)$ is convex on $[y, b)$, or*
- (B) *there is a $z \in (a, y)$ so that $g(x)/\psi(x)$ is locally increasing at z , $g'(x+) \leq g'(x-)$ for all $x \in (z, b) \cap \mathcal{P}$, and $(\mathcal{G}_r g)(x)$ is non-increasing and non-positive for all $x \in (z, b)$.*

Then, the function $\hat{f}(x)$ characterized by (17) is non-decreasing on $[y, b)$.

Proof. It is clear from (17) that the required monotonicity of \hat{f} is met provided that inequality

$$\frac{d}{dx} \left(\frac{g'(x)}{\psi'(x)} \right) < 0 \quad (20)$$

is satisfied for all $x \in [y, b) \setminus \mathcal{P}$ and

$$\hat{f}(x+) - \hat{f}(x-) = \frac{g'(x-) - g'(x+)}{\psi'(x)} > 0 \quad (21)$$

for all $x \in [y, b) \cap \mathcal{P}$. First, if g is concave and ψ is convex on $[y, b)$, then the inequalities (20) and (21) are satisfied and $g'(x)/\psi'(x)$ is non-increasing on $[y, b)$ as claimed. Assume now instead that the conditions of part (B) are satisfied. It is clear that since $[y, b) \subset (z, b)$ (21) is satisfied by assumption for all $x \in [y, b) \cap \mathcal{P}$. On the other hand, standard differentiation shows that for all $x \in (z, b) \setminus \mathcal{P}$

$$\frac{d}{dx} \left(\frac{g'(x)}{\psi'(x)} \right) = \frac{S'(x)}{\psi'^2(x)} \left[\frac{g''(x)}{S'(x)} \psi'(x) - \frac{\psi''(x)}{S'(x)} g'(x) \right] = \frac{2S'(x)\mathcal{D}(x)}{\sigma^2(x)\psi'^2(x)},$$

where

$$\mathcal{D}(x) = (\mathcal{G}_r g)(x) \frac{\psi'(x)}{S'(x)} + r(L_\psi g)(x).$$

The assumed monotonicity and non-positivity of $(\mathcal{G}_r g)(x)$ on $(z, b) \setminus \mathcal{P}$ and identity (3) now implies that

$$\begin{aligned} \mathcal{D}(x) &= (\mathcal{G}_r g)(x) \frac{\psi'(x)}{S'(x)} - r \int_z^x \psi(v) (\mathcal{G}_r g)(v) m'(v) dv + r(L_\psi g)(z+) \\ &\leq (\mathcal{G}_r g)(x) \frac{\psi'(z)}{S'(z)} + r(L_\psi g)(z+) \leq r(L_\psi g)(z+) \end{aligned}$$

for all $x \in (z, b) \setminus \mathcal{P}$. However, the assumed monotonicity of $g(x)/\psi(x)$ in a neighborhood of z then guarantees that $(L_\psi g)(z+) \leq 0$, proving that $\mathcal{D}(x) \leq 0$ for all $x \in (z, b) \setminus \mathcal{P}$. \square

Lemma 3.8 states a set of conditions under which the function \hat{f} characterized by (17) is non-decreasing on the set $[y, b)$ and, therefore, the function $\hat{f}(x)\mathbb{1}_{[y, b)}(x)$ is nondecreasing on \mathcal{I} . Interestingly, the first of these conditions is based solely on the concavity of the exercise payoff and the convexity of the increasing fundamental solution without imposing further requirements. Since the convexity of the fundamental solution ψ is determined by μ and σ , part (A) of Lemma 3.8 essentially delineates circumstances under which the monotonicity of the representing function \hat{f} could be, in principle, characterized solely based on the infinitesimal characteristics of the underlying diffusion and the concavity of the exercise payoff. Part (B) of Lemma 3.8 shows, in turn, how the monotonicity of the function \hat{f} is associated with the monotonicity of the generator $\mathcal{G}_r g$. The conditions of part (B) of Lemma 3.8 are satisfied, for example, under the assumptions of Remark 3.3 provided that $\mathcal{G}_r g$ is non-increasing on (\tilde{x}, b) and $z \in (\tilde{x}, y \wedge y^*)$.

Moreover, it is clear that under the conditions of Lemma 3.8 we have $J_y(x) = V_y(x)$ for all $x, y \in \mathcal{I}$. However, without imposing further restrictions on the behavior of the payoff we do not know whether $\hat{f}(x)\mathbb{1}_{[y, b)}(x)$ generates the smallest r -excessive majorant of the exercise payoff g or not, nor do we know how $\hat{f}(x)\mathbb{1}_{[y, b)}(x)$ behaves in the neighborhood of the optimal stopping boundary. Our next theorem summarizes a set of conditions under which these questions can be unambiguously answered.

Theorem 3.9. *Assume that there is a unique interior threshold $y^* = \inf\{x \in \mathcal{I} : \hat{f}(x) > 0\} \in \mathcal{I}$, that the conditions (A) or (B) of Lemma 3.8 are satisfied on $[y^*, b)$, and that g' is lower semicontinuous on $[y^*, b)$. Then, $\hat{f}(y^*) = 0$ if $y^* \in \mathcal{I} \setminus \mathcal{P}$ and*

$$\hat{f}(y^*) = g(y^*) - \frac{\psi(y^*)}{\psi'(y^*)} g'(y^*+) > 0$$

if $y^* \in \mathcal{P}$. Moreover,

$$\hat{f}(x) = \frac{(L_\psi g)(x+)}{(L_\psi \mathbb{1})(x)} = \frac{(L_\psi g)(y^*+) - \int_{y^*}^x (\mathcal{G}_r g)(v) \psi(v) m'(v) dv}{(L_\psi \mathbb{1})(x)} \quad (22)$$

for all $x \in (y^*, b) \setminus \mathcal{P}$, and

$$\begin{aligned} V(x) &= V_{y^*}(x) = J_{y^*}(x) = \psi(x) \sup_{y \geq x} \left[\frac{g(y)}{\psi(y)} \right] = \psi(x) \frac{g(x \vee y^*)}{\psi(x \vee y^*)} \\ &= \mathbb{E}_x \left[\sup_{t \in [0, T]} \hat{f}(X_t) \mathbb{1}_{[y^*, b)}(X_t) \right]. \end{aligned} \quad (23)$$

Proof. Claim (22) follows from the identity $\hat{f}(x) = \frac{S'(x)}{\psi'(x)}(L_\psi g)(x)$ by invoking the canonical form (3). The rest of the claims follow directly from Theorem 3.7. \square

Theorem 3.9 shows that the continuity of the function \hat{f} at the optimal boundary y^* coincides with the standard *smooth fit principle* requiring that the value should be continuously differentiable across the optimal boundary. However, as is clear from Theorem 3.9, if the optimal boundary is attained at a threshold where the exercise payoff is not differentiable, then \hat{f} is discontinuous at the optimal boundary y^* . Furthermore, since the nonnegativity and monotonicity of $\hat{f}(x) \mathbb{1}_{[y^*, b)}(x)$ on $[y^*, b)$ are sufficient for the validity of Theorem 3.9, we observe in accordance with the results by [11] that $\hat{f}(x) \mathbb{1}_{[y^*, b)}(x)$ is only upper semicontinuous on \mathcal{I} .

Theorem 3.9 also shows that $\hat{f}(x)$ has a neat integral representation (22) capturing the size of the potential discontinuity of $\hat{f}(x)$ at y^* . In the case where a is unattainable and the smooth fit principle is satisfied at y^* (22) can be re-expressed as (cf. Proposition 2.13 in [11])

$$\hat{f}(x) = - \frac{\int_{y^*}^x (\mathcal{G}_r g)(v) \psi(v) m'(v) dv}{r \int_a^x \psi(v) m'(v) dv} \quad (24)$$

and, hence, in that case the value reads as

$$V(x) = -\mathbb{E}_x \left[\frac{\int_{y^*}^{M_T} (\mathcal{G}_r g)(v) \psi(v) m'(v) dv}{r \int_a^{M_T} \psi(v) m'(v) dv} \mathbb{1}_{[y^*, b)}(M_T) \right] \quad (25)$$

It is clear that if the sufficient conditions stated in Remark 3.3 are satisfied, and in addition $(\mathcal{G}_r g)(x)$ is non-increasing on (y^*, b) , and a is unattainable for the underlying diffusion, then the conditions of Theorem 3.9 are met and

$$V(x) = \mathbb{E}_x \left[\sup_{t \in [0, T]} \left(- \frac{\int_{y^*}^{X_t} (\mathcal{G}_r g)(v) \psi(v) m'(v) dv}{r \int_a^{X_t} \psi(v) m'(v) dv} \mathbb{1}_{[y^*, b)}(X_t) \right) \right].$$

Remark 3.10. *Our approach relies on the identity*

$$\mathbb{P}_x[M_T \geq y] = \mathbb{P}_x[\tau_y < T] = \mathbb{E}_x[e^{-r\tau_y}] = \frac{\psi(x)}{\psi(y)}, \quad a < x < y < b$$

which is essentially based on the continuity of the running supremum process M_t . Since the running supremum of a spectrally negative jump-diffusion is continuous as well and a jump-diffusion is a Hunt process, we notice that our principal findings on the representing function are valid for that class of processes as well provided that a set of sufficient regularity conditions are met (cf. [2]). In that setting the increasing fundamental solution ψ can be identified as the r -scale function associated with the particular spectrally negative jump diffusion (cf. Theorem 8.1 in [30]).

4 Illustrations and Extensions

We now illustrate our general findings in five separate examples in order to illustrate the applicability of the developed approach as well as the intricacies associated with the considered representation. The first example focuses on a stopping problem arising in the literature on economic mechanism design. In the second example we reconsider the analysis of an optimal stopping signal originally studied in [4] and connect it to the analysis developed in our manuscript. The third example focuses, in turn, on a case where the payoff is smooth and the stopping strategy is of the single boundary type. Despite these favorable properties, we will show that it does not always result into a value characterizable as an expected supremum in the spirit of (15). The fourth example, in turn, focuses on a less smooth case resulting into a representation where the representing function is monotone but not everywhere continuous. Finally, the fifth example focuses on spectrally negative jump diffusions and show how the developed approach applies there as well.

4.1 Incentive Compatible Implementable Stopping Rules

[29] and [28] consider the determination of incentive compatible implementable stopping rules arising in economic studies analyzing mechanism design. One of the key questions within the framework developed in [29] and [28] is to investigate if there exists a *transfer* which would result into the optimality of a desired exercise strategy characterized by a so-called cut-off rule. Such problems arise quite naturally, for example, in models considering situations where individual exercise strategies do not coincide with a socially desirable exercise rule. In

such cases the decision making problem of a social planner can be reduced into the determination of a transfer rule (for example, a tax) resulting into the individual optimality of the socially desirable state. As we will now demonstrate, the approach developed in this study is particularly appropriate for the analysis of this question within the considered infinite horizon setting. To see that this is indeed the case, assume for simplicity that the exercise payoff $g : \mathcal{I} \mapsto \mathbb{R}$ is continuously differentiable on \mathcal{I} . Assume also that the representing function \hat{f} defined for $x \in \mathcal{I}$ by $\hat{f}(x) = g(x) - g'(x)\psi(x)/\psi'(x)$ is nondecreasing and changes uniquely sign at the interior threshold $y^* = \hat{f}^{-1}(0) \in (a, b)$. It is clear from Theorem 3.9 that in that case $y^* = \operatorname{argmax}\{g(x)/\psi(x)\}$, $\tau_{y^*} = \inf\{t \geq 0 : X_t \geq y^*\}$ is an optimal stopping time, and the value reads as in (23). Given these observations, we now consider the associated optimal stopping problem

$$V^{\hat{f}}(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-r\tau} (g(X_\tau) - \hat{f}(k^*)) \right]. \quad (26)$$

where $k^* \in \mathcal{I}$ is an exogenously set threshold. We now find that our assumptions guarantee the following result.

Proposition 4.1. *Under our assumptions,*

$$k^* = \operatorname{argmax} \left\{ \frac{g(x) - f(k^*)}{\psi(x)} \right\},$$

$\tau_{k^*} = \inf\{t \geq 0 : X_t \geq k^*\}$ is an optimal stopping time, and the value reads as

$$\begin{aligned} V^{\hat{f}}(x) &= \mathbb{E}_x \left[e^{-r\tau_{k^*}} (g(X_{\tau_{k^*}}) - \hat{f}(k^*)) \right] = \psi(x) \frac{g(x \vee k^*) - \hat{f}(k^*)}{\psi(x \vee k^*)} \\ &= \mathbb{E}_x \left[\sup_{t \in [0, T]} \left(\hat{f}(X_t \vee k^*) - \hat{f}(k^*) \right) \right]. \end{aligned} \quad (27)$$

Proof. Consider now the function $f(x) := \hat{f}(x) - \hat{f}(k^*)$. It is clear that our assumptions guarantee that f vanishes at k^* and is continuous and nondecreasing on \mathcal{I} . On the other hand, the representing function of the exercise payoff $\hat{g}(x) := g(x) - \hat{f}(k^*)$ reads as

$$\hat{g}(x) - \hat{g}'(x) \frac{\psi(x)}{\psi'(x)} = \hat{f}(x) - \hat{f}(k^*) = f(x).$$

The alleged result now follows from Theorem 3.9. \square

Proposition 4.1 states a set of conditions under which the representing function \hat{f} can be utilized for characterizing a transfer rule resulting

into the optimality of a desired fixed exercise threshold. This result is interesting since it essentially delineates circumstances under which a social planner can shift the individually optimal exercise threshold of a decision maker to a different socially optimal level by simply subtracting (or adding) the value of the representing function at the desired state to the exercise payoff. As we will notice in the following section, this result is closely related with the literature on optimal stopping signals and Gittins indices as well. A nice comparative static implication of our Lemma 2.3 is summarized in the following.

Corollary 4.2. *Assume that there is a $\tilde{y} \in [a, b]$ so that $\mu(x) - rx$ is non-increasing on (a, \tilde{y}) , $\mu(x) \leq 0$ for $x \in (\tilde{y}, b)$ and $\lim_{x \downarrow a} \mu(x) \leq 0$ whenever a is attainable for the underlying diffusion. Then increased volatility decreases or leaves unchanged the transfer $\hat{f}(k^*)$.*

Proof. The alleged result is a direct consequence of Lemma 2.3 after noticing that the assumptions guarantee that ψ is strictly convex on \mathcal{I} (see Lemma 3.3 in [3]). \square

4.2 Optimal Stopping Signals

We now proceed in our illustrations and show how the developed representation approach is related with non-standard stopping problems arising in the analysis of Gittins indices and optimal stopping signals (cf., for example, [4], [6], [15], [16], [27], and [32]). In line with the notation in [4], we assume that $k \in \mathbb{R}$ is an exogenously given parameter and let

$$V_k(x) := \sup_{\tau} \mathbb{E}_x [e^{-r\tau} (g(X_\tau) - k)] \quad (28)$$

denote the value of the considered optimal stopping problem and assume that the exercise payoff is continuously differentiable on \mathcal{I} . As in [4] we also assume that the boundaries are natural for the underlying diffusion X . This guarantees that even though the process may tend towards a boundary, it will never attain it in finite time. In connection with problem (28), we also consider the associated non-standard stopping problem

$$\gamma(x) = \inf_{\tau \in \mathcal{T}} \frac{\mathbb{E}_x [g(x) - e^{-r\tau} g(X_\tau)]}{1 - \mathbb{E}_x [e^{-r\tau}]}, \quad (29)$$

where \mathcal{T} denotes the class of first exit times from open subsets of \mathcal{I} with compact closure in \mathcal{I} . As was established in [4], the stopping region $\Gamma_k = \{x \in \mathcal{I} : V_k(x) = g(x) - k\}$ for the problem (28) coincides with the set $\{x \in \mathcal{I} : \gamma(x) \geq k\}$. We now plan to show how these results can be replicated by utilizing the representation result developed in our paper.

It is clear from our analysis that in the present case it is sufficient that for $k \in \mathbb{R}$ the representing function

$$\hat{f}_k(x) = g(x) - k - \psi(x) \frac{g'(x)}{\psi'(x)} = \hat{f}_0(x) - k \quad (30)$$

is nondecreasing and changes uniquely sign at the interior threshold $y_k^* = \hat{f}_k^{-1}(0) \in (a, b)$ satisfying equation $\hat{f}_0(y_k^*) = k$. If that is the case, then

$$V_k(x) = \mathbb{E}_x \left[\sup_{t \in [0, T]} \hat{f}_k(X_t \vee y_k^*) \right] = \mathbb{E}_x \left[\sup_{t \in [0, T]} \left(\hat{f}_0(X_t \vee y_k^*) - k \right) \right]$$

and $\Gamma_k = \{x \in \mathcal{I} : \hat{f}_0(x) \geq k\}$. As was established in Theorem 13 of [4], $\gamma(x) = \hat{f}_0(x)$ in the present single boundary setting (see also Section 3.10 in [16] for the decreasing case). Consequently, we notice that the considered supremum representation results in the correct expressions for the considered functionals. Moreover, given the identity $\gamma(x) = \hat{f}_0(x)$ we observe the following interesting comparative static property of the the value (29):

Corollary 4.3. *Assume that the condition of Corollary 4.2 are satisfied. Then increased volatility increases the value (29).*

Proof. Analogous with the proof of Corollary 4.2. \square

We would like to point out at that the determination of incentive compatible implementable stopping rules considered in the previous subsection is closely associated with the present case as well. To see that this is indeed the case, we immediately notice that if the representing function \hat{f}_0 is monotonically increasing then choosing $k = \hat{f}_0(z^*)$ for some fixed threshold $z^* \in \mathcal{I}$ implies that $\Gamma_{\hat{f}_0(z^*)} = \{x \in \mathcal{I} : \gamma(x) \geq \hat{f}_0(z^*)\} = \{x \in \mathcal{I} : \hat{f}_0(x) \geq \hat{f}_0(z^*)\} = [z^*, b)$.

Finally, it is also worth noticing that the non-standard stopping problem (29) has in many cases an interesting interpretation as an appropriate maximal conditional expectation. To see that this is indeed the case, denote by Θ the set of functions $g : \mathcal{I} \mapsto \mathbb{R}$ belonging into the domain of the extended operator of the underlying process X killed at T and satisfying for $\tau \in \mathcal{T}$ the generalized Dynkin formula (see, for example, [11], [13], [25], and [31])

$$\mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right] = g(x) + \mathbb{E}_x \left[\int_0^\tau e^{-rs} \tilde{g}(X_s) ds \right], \quad (31)$$

where $\tilde{g} \in \mathcal{L}_r^1(\mathcal{I})$ naturally coincides with the generator $(\mathcal{G}_r g)(x)$ whenever the payoff is sufficiently smooth. It is now clear that in this case

(29) can be re-expressed as

$$\gamma(x) = -\sup_{\tau \in \mathcal{T}} \frac{\mathbb{E}_x \int_0^\tau e^{-rs} \tilde{g}(X_s) ds}{r \mathbb{E}_x \int_0^\tau e^{-rs} ds} = -\sup_{\tau \in \mathcal{T}} \frac{1}{r} \mathbb{E}_x [\tilde{g}(X_T) | T < \tau]. \quad (32)$$

Especially, if the exercise payoff constitutes an expected cumulative present value of a flow and reads as $g(x) = (R_r \pi)(x)$ for some continuous $\pi \in \mathcal{L}_r^1(\mathcal{I})$ then $g \in \Theta$ and the non-standard stopping problem (29) can be re-expressed in a more familiar form as

$$\gamma(x) = \sup_{\tau \in \mathcal{T}} \frac{\mathbb{E}_x \int_0^\tau e^{-rs} \pi(X_s) ds}{r \mathbb{E}_x \int_0^\tau e^{-rs} ds}$$

implying along the lines of our Lemma 2.2 that

$$\gamma(x) = \sup_{\tau \in \mathcal{T}} \frac{1}{r} \mathbb{E}_x [\pi(X_T) | T < \tau] = \hat{f}_0(x). \quad (33)$$

Consequently, we notice that in this case the representing function can be interpreted as the maximal expected present value of the cash flow at the independent exponential terminal date provided that the process is still alive at that instant.

4.3 Optimal Entry

In order to illustrate circumstances where the value of a single boundary problems cannot necessarily be expressed as an expected supremum, we now assume that the upper boundary b is unattainable for X and that the exercise payoff can be expressed as an expected cumulative present value $g(x) = (R_r \pi)(x)$ for some continuous revenue flow $\pi \in \mathcal{L}_r^1(\mathcal{I})$ satisfying the conditions $\pi(x) \geq 0$ for $x \geq x_0$, where $x_0 \in (a, b)$, $\lim_{x \downarrow a} \pi(x) < -\varepsilon$ and $\lim_{x \uparrow b} \pi(x) > \varepsilon$ for some $\varepsilon > 0$. This type of models arise frequently in studies considering optimal entry under uncertainty.

It is clear that under these conditions the exercise payoff satisfies the conditions $g \in C^2(\mathcal{I})$ and $(\mathcal{G}_r g)(x) = -\pi(x) \leq 0$ for $x \geq x_0$. Moreover, utilizing representation (4) shows that in the present case

$$(L_\psi g)(x) = \int_a^x \psi(t) \pi(t) m'(t) dt$$

Our assumptions guarantee that $(L_\psi g)(x) < 0$ for all $x \leq x_0$ and that $(L_\psi g)(x)$ is monotonically increasing on (x_0, b) . Fix $x_1 > x_0$. Then a standard application of the mean value theorem for definite integrals

yields

$$\begin{aligned}(L_\psi g)(x) &= (L_\psi g)(x_1) + \int_{x_1}^x \psi(t)\pi(t)m'(t)dt \\ &= (L_\psi g)(x_1) + \frac{\pi(\xi)}{r} \left[\frac{\psi'(x)}{S'(x)} - \frac{\psi'(x_1)}{S'(x_1)} \right],\end{aligned}$$

where $\xi \in (x_1, x)$. Letting $x \rightarrow b$ and noticing that $\psi'(x)/S'(x) \rightarrow \infty$ as $x \rightarrow b$ (since b was assumed to be unattainable for X , cf. p. 19 in [8]) then shows that $\lim_{x \uparrow b} (L_\psi g)(x) = \infty$ proving that equation $(L_\psi g)(x) = 0$ has a unique root $y^* \in (x_0, b)$ and that $y^* = \operatorname{argmax}\{(R_r\pi)(x)/\psi(x)\}$. Moreover, the value (6) can be expressed as

$$V(x) = \psi(x) \frac{(R_r\pi)(x \vee y^*)}{\psi(x \vee y^*)} = \begin{cases} (R_r\pi)(x) & x \geq y^* \\ \frac{(R_r\pi)(y^*)}{\psi(y^*)} \psi(x) & x < y^*. \end{cases}$$

The representing function $\hat{f}(x)$ characterized in Theorem 3.4 can be expressed in the present setting as

$$\hat{f}(x) = \frac{S'(x)}{\psi'(x)} \int_a^x \psi(y)\pi(y)m'(y)dy.$$

As was established in Theorem 3.9, we have that $\hat{f}(y^*) = 0$ and

$$V(x) = \mathbb{E}_x \left[\frac{S'(M_T)}{\psi'(M_T)} \int_{y^*}^{M_T \vee y^*} \psi(y)\pi(y)m'(y)dy \right].$$

Moreover, standard differentiation shows that for all $x \in (y^*, b)$ we have

$$\hat{f}'(x) = \frac{2S'(x)\psi(x)}{\psi'^2(x)\sigma^2(x)} \left[\pi(x) \frac{\psi'(x)}{S'(x)} - r \int_{y^*}^x \psi(t)\pi(t)m'(t)dt \right]$$

demonstrating that \hat{f} is nondecreasing for $x \in (y^*, b)$ only if

$$\pi(x) \frac{\psi'(x)}{S'(x)} \geq r \int_{y^*}^x \psi(t)\pi(t)m'(t)dt$$

for all $x \geq y^*$. Otherwise it is clear from our results that the value of the considered optimal stopping problem *cannot* be expressed as an expected supremum of the form (15) (see Figure 1(a)). A simple sufficient condition guaranteeing the required monotonicity is to assume that $\pi(x)$ is nondecreasing on (x_0, b) since in that case we have

$$\begin{aligned}\hat{f}'(x) &\geq \frac{2S'(x)\psi(x)}{\psi'^2(x)\sigma^2(x)} \left[\pi(x) \frac{\psi'(x)}{S'(x)} - r\pi(x) \int_{y^*}^x \psi(t)m'(t)dt \right] \\ &\geq \frac{2S'(x)\psi(x)}{\psi'^2(x)\sigma^2(x)} \pi(x) \frac{\psi'(y^*)}{S'(y^*)} \geq 0.\end{aligned}$$

If this is indeed the case, then

$$V(x) = \mathbb{E}_x \left[\sup_{t \in [0, T]} \frac{S'(X_t \vee y^*)}{\psi'(X_t \vee y^*)} \int_{y^*}^{X_t \vee y^*} \psi(v) \pi(v) m'(v) dv \right].$$

4.4 Capped Call Option

In order to illustrate our findings in a nondifferentiable setting, assume now that the upper boundary b is unattainable for X and that the exercise payoff $g(x) = \min((x - K)^+, C)$, with $a < K < K + C < b$, satisfies the limiting inequality

$$\lim_{x \downarrow a} \frac{|x - K|}{\varphi(x)} < \infty. \quad (34)$$

Assume also that the appreciation rate $\theta(x) = \mu(x) - r(x - K)$ satisfies the conditions $\theta \in \mathcal{L}_r^1(\mathcal{I})$, $\theta(x) \geq 0$ for $x \geq x_0^\theta$, where $x_0^\theta \in \mathcal{I}$, and $\lim_{x \rightarrow b} \theta(x) < -\varepsilon$ for $\varepsilon > 0$.

We notice that the exercise payoff g is continuous, nondecreasing, and twice continuously differentiable on $\mathcal{I} \setminus \{K, K + C\}$. Moreover, $g'(K-) \leq g'(K+)$, $\lim_{x \rightarrow (K+C)-} g'(x) \geq \lim_{x \rightarrow (K+C)+} g'(x)$, and

$$(\mathcal{G}_r g)(x) = \begin{cases} -rC, & x \in (K + C, b) \\ \theta(x), & x \in (K, K + C) \\ 0, & x \in (a, K). \end{cases}$$

It is now clear that the conditions of Remark 3.3 are satisfied. Thus, we know that there exists a unique optimal exercise threshold $x^* = \operatorname{argmax}\{g(x)/\psi(x)\}$ and $V(x) = V_{x^*}(x)$. Our objective is now to prove that this threshold reads as $x^* = \min(C + K, y^*)$, where $y^* > x_0^\theta$ is the unique root of the ordinary first order condition

$$\psi(y^*) = \psi'(y^*)(y^* - K).$$

To see that this is indeed the case, we first observe by applying part (A) of Corollary 3.2 in [1] combined with the limiting condition (34) that

$$\frac{\psi^2(x)}{S'(x)} \frac{d}{dx} \left[\frac{x - K}{\psi(x)} \right] = \frac{\psi(x)}{S'(x)} - (x - K) \frac{\psi'(x)}{S'(x)} = \int_a^x \psi(t) \theta(t) m'(t) dt - \frac{a - K}{\varphi(a)}.$$

Applying analogous arguments with the ones in Example 3, we find that equation

$$\int_a^x \psi(t) \theta(t) m'(t) dt - \frac{a - K}{\varphi(a)} = 0$$

has a unique root $y^* \in (x_0^\theta, b)$ so that $y^* = \operatorname{argmax}\{(x - K)/\psi(x)\}$. Moreover,

$$U(x) = \sup_{\tau} \mathbb{E}_x [e^{-r\tau} (X_\tau - K)^+] = \begin{cases} x - K & x \geq y^* \\ (y^* - K) \frac{\psi(x)}{\psi(y^*)} & x < y^*. \end{cases}$$

In light of these observations, we find that if $y^* \in (K, K + C)$, then it is sufficient to notice that $V_{x^*}(x) = \min(C, U(x))$ is r -excessive since constants are r -excessive and $U(x)$ is also r -excessive. Moreover, since both C and $U(x)$ dominate the payoff, we notice that $V_{x^*}(x) = \min(C, U(x))$ constitutes the smallest r -excessive majorant of $g(x)$ and, therefore, $V(x) = V_{x^*}(x) = \min(C, U(x))$. If instead $y^* \geq K + C$, then $x^* = K + C = \operatorname{argmax}\{g(x)/\psi(x)\}$ and the optimal policy is to follow the stopping policy $\tau_{x^*} = \inf\{t \geq 0 : X_t \geq K + C\}$ with a value

$$\tilde{U}(x) = C \mathbb{E}_x [e^{-r\tau_{x^*}}] = \begin{cases} C & x \geq C + K \\ C \frac{\psi(x)}{\psi(C+K)} & x < C + K. \end{cases}$$

Given these findings, we notice that if $y^* \geq K + C$, then $x^* = K + C$ and

$$f(x) = C \mathbb{1}_{[x^*, b)}(x) \geq 0$$

is nonnegative and nondecreasing and, consequently,

$$V(x) = C \mathbb{E}_x [\mathbb{1}_{[x^*, b)}(M_T)] = C \mathbb{P}_x [M_T \geq K + C].$$

However, since $f(x^* -) = 0$ and $f(x^* +) = C$ we notice that f is discontinuous at the optimal threshold x^* (see Figure 1(b)). If $y^* < K + C$, then the nonnegative function

$$f(x) = \begin{cases} C & x \geq C + K \\ x - K - \frac{\psi(x)}{\psi'(x)} & x \in [y^*, K + C) \end{cases}$$

is nondecreasing only if the increasing fundamental solution is convex on $(y^*, K + C)$ (it has to be locally convex at y^*). If the convexity requirement is met, then

$$V(x) = \mathbb{E}_x \left[\left(M_T - K - \frac{\psi(M_T)}{\psi'(M_T)} \right) \mathbb{1}_{[y^*, C+K)}(M_T) \right] + C \mathbb{P}_x [M_T \geq C + K].$$

Moreover, since $f(C + K +) = C > C - \frac{\psi(C+K-)}{\psi'(C+K-)} = f(C + K -)$, we notice that f is discontinuous at $C + K$.

4.5 Spectrally Negative Jump-diffusions

In order to illustrate our findings for a spectrally negative jump diffusion, consider now the geometric Lévy process $X = \{X_t\}$ with a finite

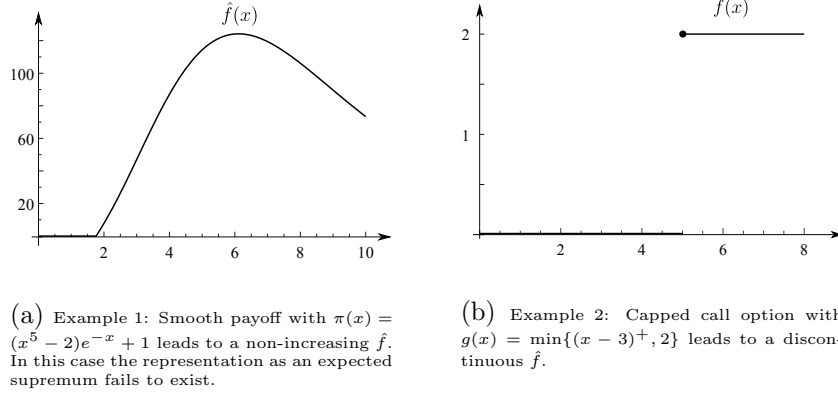


Figure 1: Numerical examples based on geometric Brownian motion. Parameters have been chosen such that $\psi = x^2$ and $\varphi = x^{-4}$

Lévy measure $\nu = \lambda \mathbf{m}$, where \mathbf{m} denotes the jump size distribution, characterized by the dynamics

$$dX_t = X_{t-} \left\{ \mu dt + \sigma dW_t + \lambda \int_{(0,1)} z \tilde{N}(dt, dz) \right\}, \quad X_0 := x \in \mathbb{R}_+$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. It is now a straightforward exercise to show that $\psi(x) = x^\varrho$, where $\varrho > 0$ denotes the positive root of the characteristic equation

$$\frac{1}{2} \sigma^2 \varrho(\varrho - 1) + (\mu + \lambda \bar{m}) \varrho - (r + \lambda) + \lambda \int_0^1 (1 - z)^\varrho \mathbf{m}(dz) = 0,$$

and

$$\bar{m} = \int_0^1 z \mathbf{m}(dz)$$

denotes the expected jump size. We observe that if

$$\hat{f}(x) = g(x) - \frac{1}{\varrho} g'(x) x$$

is nondecreasing, there is an interior point $y^* = \inf\{x \geq 0 : \hat{f}(x) \geq 0\} \in (0, \infty)$, and g' is lower semicontinuous on $[y^*, \infty)$, then

$$V(x) = \mathbb{E}_x \left[\sup\{\hat{f}(X_t) \mathbb{1}_{[y^*, \infty)}(X_t); t \leq T\} \right] = \begin{cases} g(x), & x \in [y^*, \infty), \\ g(y^*) \left(\frac{x}{y^*} \right)^\varrho, & x \in (0, y^*). \end{cases}$$

It is at this point worth emphasizing that in this jump-diffusion setting verifying optimality by investigating the behavior of the representing

function \hat{f} is easier than by investigating the behavior of the generator of the underlying process.

5 Conclusions

We considered the representation of the value of a class of optimal stopping problems of linear diffusions as the expected supremum of a function with known regularity and monotonicity properties. By focusing on the single exercise boundary case, we developed an explicit integral representation for the above mentioned function by first computing the probability distribution of the running supremum of the underlying diffusion and then utilizing this distribution in determining the expected value explicitly in terms of the increasing minimal excessive mapping and the infinitesimal characteristics of the diffusion.

There are at least three directions towards which our analysis could be potentially extended. First, the present approach focuses on single boundary problems and consequently overlooks general problems with more boundaries. Extending our analysis towards this setting and computing the representing function explicitly would, therefore, constitute a natural extension of our approach. Second, impulse control and optimal switching problems can in many diffusion cases be interpreted as sequential stopping problems of the underlying process. Thus, extending our representation to that setting would be interesting too (for a recent approach to this within impulse control, see [10]). However, given the potential discreteness of the optimal policy in the impulse control policy setting seems to make the explicit determination of the integral representation a very challenging problem which at the moment is outside the scope of our study.

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References

- [1] L. H. R. Alvarez E., *A class of solvable stopping games*, Appl. Math. Optim. **58** (2008), no. 3, 291–314.
- [2] L. H. R. Alvarez E., P. Matomäki, and T. A. Rakkolainen, *A class*

of solvable optimal stopping problems of spectrally negative jump diffusions, SIAM J. Control Optim. **52** (2014), no. 4, 2224–2249.

- [3] Luis H. R. Alvarez E., *A class of solvable impulse control problems*, Appl. Math. Optim. **49** (2004), no. 3, 265–295.
- [4] P. Bank and C. Baumgarten, *Parameter-dependent optimal stopping problems for one-dimensional diffusions*, Electron. J. Probab. **15** (2010), 1971–1993.
- [5] P. Bank and N. El Karoui, *A stochastic representation theorem with applications to optimization and obstacle problems*, Ann. Probab. **32** (2004), 1030–1067.
- [6] P. Bank and H. Föllmer, *American options, multi-armed bandits, and optimal consumption plans: a unifying view*, Paris-Princeton Lectures on Mathematical Finance, 2002, Lecture Notes in Math., vol. 1814, Springer, Berlin, 2003, pp. 1–42.
- [7] P. Bank and F. Riedel, *Optimal consumption choice with intertemporal substitution*, Ann. Appl. Probab. **11** (2001), 750–788.
- [8] A. N. Borodin and P. Salminen, *Handbook of Brownian motion—facts and formulae*, (corrected second printing) second ed., Probability and its Applications, Birkhäuser Verlag, Basel, 2015.
- [9] M. B. Chiarolla and G. Ferrari, *Identifying the free boundary of a stochastic, irreversible investment problem via the Bank–El Karoui representation theorem*, SIAM J. Control Optim. **52** (2014), no. 2, 1048–1070.
- [10] S. Christensen and P. Salminen, *Impulse control and expected suprema*, arXiv:1503.01253 (2015).
- [11] S. Christensen, P. Salminen, and B. Q. Ta, *Optimal stopping of strong markov processes*, Stochastic Process. Appl. **123** (2013), 1138–1159.
- [12] Sören Christensen and Albrecht Irle, *A harmonic function technique for the optimal stopping of diffusions*, Stochastics **83** (2011), no. 4-6, 347–363.
- [13] F. Croce and E. Mordecki, *Explicit solutions in one-sided optimal stopping problems for one-dimensional diffusions*, Stochastics **86** (2014), no. 3, 491–509.
- [14] T. De Angelis, G. Ferrari, and J. Moriarty, *A Nonconvex Singular Stochastic Control Problem and its Related Optimal Stopping Boundaries*, SIAM J. Control Optim. **53** (2015), no. 3, 1199–1223.

- [15] N. El Karoui and H. Föllmer, *A non-linear riesz representation in probabilistic potential theory*, Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), 269–283.
- [16] N. El Karoui and I. Karatzas, *Dynamic allocation problems in continuous time*, Ann. Appl. Probab. **4** (1994), no. 2, 255–286.
- [17] N. El Karoui and A. Meziou, *Max-plus decomposition of supermartingales and convex order. application to american options and portfolio insurance*, Ann. Probab. **36** (2008), 647–697.
- [18] G. Ferrari, *On an integral equation for the free-boundary of stochastic, irreversible investment problems*, Ann. Appl. Probab. **25** (2015), no. 1, 150–176.
- [19] G. Ferrari and P. Salminen, *Irreversible investment under lévy uncertainty: an equation for the optimal boundary*, Advances in Applied Probability (to appear) (2014).
- [20] H. Föllmer and T. Knispel, *A representation of excessive functions as expected suprema*, Probab. Math. Statist. **26** (2006), 379–394.
- [21] ———, *Potentials of a markov process are expected suprema*, ESAIM: Probabability and Statistics **11** (2007), 89–101.
- [22] J. C. Gittins, *Bandit processes and dynamic allocation indices*, Journal of the Royal Statistical Society. Series B **41** (1979), no. 2, 148–177, With discussion.
- [23] J. C. Gittins and K. D. Glazebrook, *On Bayesian models in stochastic scheduling*, J. Appl. Probab. **14** (1977), no. 3, 556–565.
- [24] J. C. Gittins and D. M. Jones, *A dynamic allocation index for the discounted multiarmed bandit problem*, Biometrika **66** (1979), no. 3, 561–565.
- [25] K. Helmes and R. H. Stockbridge, *Construction of the value function and optimal rules in optimal stopping of one-dimensional diffusions*, Adv. in Appl. Probab. **42** (2010), no. 1, 158–182.
- [26] I. Karatzas, *Gittins indices in the dynamic allocation problem for diffusion processes*, Ann. Probab. **12** (1984), no. 1, 173–192.
- [27] Haya Kaspí and Avishai Mandelbaum, *Multi-armed bandits in discrete and continuous time*, Ann. Appl. Probab. **8** (1998), no. 4, 1270–1290.
- [28] Thomas Kruse and Philipp Strack, *Inverse optimal stopping*, arXiv:1406.0209v2 (2015).

- [29] ———, *Optimal stopping with private information*, J. Econom. Theory **159** (2015), no. part B, 702–727.
- [30] Andreas E. Kyprianou, *Introductory lectures on fluctuations of Lévy processes with applications*, Universitext, Springer-Verlag, Berlin, 2006.
- [31] D. Lamberton and M. Zervos, *On the optimal stopping of a one-dimensional diffusion*, Electron. J. Probab. **18** (2013), no. 34, 49.
- [32] Avi Mandelbaum, *Continuous multi-armed bandits and multiparameter processes*, Ann. Probab. **15** (1987), no. 4, 1527–1556.
- [33] P. Salminen, *Optimal stopping of one-dimensional diffusions*, Math. Nachr. **124** (1985), 85–101.

A Proof of Lemma 2.2

Proof. (A) Assume that $h : \mathcal{I} \mapsto \mathbb{R}$ is such that $h \in \mathcal{L}_r^1(\mathcal{I})$. Noticing that

$$\mathbb{P}_x[X_T \in dv | M_T \leq y] = \frac{\mathbb{P}_x[X_T \in dv; M_T \leq y]}{\mathbb{P}_x[M_T \leq y]} = \frac{\mathbb{P}_x[X_T \in dv; T < \tau_y]}{\mathbb{P}_x[T < \tau_y]}$$

demonstrates that

$$\frac{1}{r} \mathbb{E}_x[h(X_T) | M_T \leq y] = \frac{\mathbb{E}_x \int_0^{\tau_y} e^{-rs} h(X_s) ds}{1 - \mathbb{E}_x[e^{-r\tau_y}]}.$$

Invoking the strong Markov property and utilizing the known form of the Laplace transform of the first hitting time τ_y (p. 18 on [8]) yields (8). On the other hand, combining the joint probability density $\mathbb{P}_x[X_T \in dv, M_T \in dy]$ of X and M stated on p. 26 of [8] with the density $\mathbb{P}_x[M_T \in dy] = (\psi'(y)\psi(x)/\psi^2(y))dy$ yields

$$\mathbb{P}_x[X_T \in dv | M_T = y] = r \frac{S'(y)}{\psi'(y)} \psi(v) m'(v) dv, \quad x \in (a, y].$$

The proposed expectation (9) then follows by standard integration. Letting $x \uparrow y$ in (8) and invoking L'Hospital's rule yields

$$\lim_{x \uparrow y} \frac{1}{r} \mathbb{E}_x[h(X_T) | M_T \leq y] = (R_r h)(y) - (R_r h)'(y) \frac{\psi(y)}{\psi'(y)}.$$

Applying now the representation (4) yields the proposed identity (10), thus completing the proof of part (A). (B) Finally, identity (11) follows under the assumption of the lemma from (3) and (9) after letting $y \downarrow a$ and noticing that $(L_\psi \mathbb{1})(y) = \psi'(y)/S'(y)$. \square

B Proof of Lemma 2.3

Proof. In order to prove the alleged comparative static results, we first denote by $\tilde{\psi}$ the increasing fundamental solution associated with the more volatile dynamics characterized by the volatility coefficient $\tilde{\sigma}(x) \geq \sigma(x)$ for all $x \in \mathcal{I}$. Denote by $\tau_{(y_0, y_1)} = \inf\{t \geq 0 : X_t \notin (y_0, y_1)\}$ the first exit date of the diffusion X from the open subset $(y_0, y_1) \subset \mathcal{I}$, where $a < y_0 < y_1 < b$. Standard application of Dynkin's theorem shows that for all $x \in (y_0, y_1)$ we have

$$\mathbb{E}_x \left[e^{-r\tau_{(y_0, y_1)}} \tilde{\psi}(X_{\tau_{(y_0, y_1)}}) \right] = \tilde{\psi}(x) + \mathbb{E}_x \int_0^{\tau_{(y_0, y_1)}} e^{-rs} (\mathcal{G}_r \tilde{\psi})(X_s) ds \leq \tilde{\psi}(x),$$

since

$$(\mathcal{G}_r \tilde{\psi})(x) = \frac{1}{2}(\sigma^2(x) - \tilde{\sigma}^2(x))\tilde{\psi}''(x) \leq 0$$

by the assumed convexity of $\tilde{\psi}$. Consequently, by utilizing standard computations we notice that for all $x \in (y_0, y_1)$ it holds

$$\begin{aligned} \tilde{\psi}(x) &\geq \tilde{\psi}(y_0) \mathbb{E}_x [e^{-r\eta_{y_0}}; \eta_{y_0} < \eta_{y_1}] + \tilde{\psi}(y_1) \mathbb{E}_x [e^{-r\eta_{y_1}}; \eta_{y_0} > \eta_{y_1}] \\ &\geq \tilde{\psi}(y_1) \mathbb{E}_x [e^{-r\eta_{y_1}}; \eta_{y_0} > \eta_{y_1}] = \tilde{\psi}(y_1) \frac{\psi(x) - \varphi(x) \frac{\psi(y_0)}{\varphi(y_0)}}{\psi(y_1) - \varphi(y_1) \frac{\psi(y_0)}{\varphi(y_0)}} \end{aligned}$$

where $\eta_z = \inf\{t \geq 0 : X_t = z\}$ denotes the first hitting time of X to a state $z \in \mathcal{I}$. Letting $y_0 \downarrow a$ and utilizing the fact that $\lim_{x \downarrow a} \psi(x)/\varphi(x) = 0$ for the considered class of boundary behaviors (cf. [8], p. 19) then shows that

$$\frac{\tilde{\psi}(x)}{\tilde{\psi}(y_1)} \geq \frac{\psi(x)}{\psi(y_1)}$$

for all $x \in (a, y_1)$. On the other hand, noticing that

$$\frac{\psi(x)}{\psi(y_1)} = \exp \left(- \int_x^{y_1} \frac{\psi'(t)}{\psi(t)} dt \right) \leq \exp \left(- \int_x^{y_1} \frac{\tilde{\psi}'(t)}{\tilde{\psi}(t)} dt \right) = \frac{\tilde{\psi}(x)}{\tilde{\psi}(y_1)}$$

for all $a < x \leq y_1 < b$ implies that $\psi'(x)/\psi(x) \geq \tilde{\psi}'(x)/\tilde{\psi}(x)$ for all $x \in \mathcal{I}$, thus completing the proof of our lemma. \square

C Proof of Lemma 3.2

Proof. It is clear that under our assumptions $V_{y^*}(x)$ is nonnegative, continuous, and dominates the exercise payoff $g(x)$ for all $x \in \mathcal{I}$. Moreover, since $y^* \in \{x \in \mathcal{I} : V(x) = g(x)\}$ by Theorem 2.1 in [12], we

find that the stopping region is nonempty. Let $x_0 \in (y^*, b) \setminus \mathcal{P}$ be a fixed reference point and define the ratio $h_{x_0}(x) = V_{y^*}(x)/V_{y^*}(x_0) = V_{y^*}(x)/g(x_0)$. It is clear that our assumptions combined with (2) guarantee that

$$\sigma_{x_0}^{h_{x_0}}((x, b]) = \frac{\psi(x_0)}{Bg(x_0)} \left[\frac{g'(x+)}{S'(x)} \varphi(x) - g(x) \frac{\varphi'(x)}{S'(x)} \right] = -\frac{\psi(x_0)}{Bg(x_0)} (L_\varphi g)(x+)$$

is nonnegative and nonincreasing for all $x \geq x_0$. Analogously,

$$\begin{aligned} \sigma_{x_0}^{h_{x_0}}([a, x)) &= \frac{\varphi(x_0)}{Bg(x_0)} \left[g(x) \frac{\psi'(x)}{S'(x)} - \frac{g'(x-)}{S'(x)} \psi(x) \right] \mathbb{1}_{(y^*, x_0]}(x) \\ &= \frac{\varphi(x_0)}{Bg(x_0)} (L_\psi g)(x-) \mathbb{1}_{(y^*, x_0]}(x) \end{aligned}$$

is nonnegative and nondecreasing for all $x \leq x_0$. Moreover, noticing that $\sigma_{x_0}^{h_{x_0}}([a, x_0)) + \sigma_{x_0}^{h_{x_0}}((x_0, b]) = 1$ shows, by imposing the condition $\sigma_{x_0}^{h_{x_0}}(\{x_0\}) = 0$, that $\sigma_{x_0}^{h_{x_0}}$ constitutes a probability measure. Therefore, it induces an r -excessive function $h_{x_0}(x)$ via its Martin representation (cf. Proposition 3.3 in [33]). However, since increasing linear transformations of excessive functions are excessive and $h_{x_0}(x)g(x_0) = V_{y^*}(x)$, we observe that $V_{y^*}(x)$ constitutes an r -excessive majorant of g for X . Invoking now (13) shows that $V(x) = V_{y^*}(x)$ and consequently, that $\tau_{y^*} = \inf\{t \geq 0 : X_t \geq y^*\}$ is an optimal stopping time. \square